Comprehensive Two-Dimensional Stress Analysis

Plane Stress, Plane Strain, and Axisymmetric Elasticity

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1 Introduction

Three-dimensional elasticity provides the most general description of stress and deformation in a solid. In a general body, displacement, strain, and stress may vary in all three spatial directions, and six independent stress components may be needed to describe the local state. A full three-dimensional model is therefore the natural choice when geometry, loading, boundary conditions, or material behavior are genuinely three-dimensional.

Many engineering problems, however, contain strong geometric or physical structure. A sheet may be very thin and loaded mainly in its plane. A dam, tunnel, or retaining wall may extend a long distance with little change along that direction. A pressure vessel, seal, or shaft may be generated by revolving one cross-section about an axis. In such cases, selected features of the three-dimensional response may be absent or negligible, allowing the problem to be represented by a two-dimensional formulation.

Two-dimensional analysis is valuable because it reduces the number of spatial variables while preserving the dominant mechanics. The geometry becomes easier to describe, calculations become more transparent, and parametric studies become much faster. Just as importantly, a carefully chosen 2D model often provides more physical insight than a large 3D calculation because the assumptions and the active stress components are visible explicitly.

The reduction is not based on geometry alone. A thin component is not automatically a plane-stress problem; transverse pressure may cause bending that a 2D plane-stress continuum cannot represent. A long body is not automatically in plane strain; it may be free to expand longitudinally. A round component is not automatically axisymmetric; one side hole, discrete support, or eccentric force may destroy circumferential invariance. A 2D formulation is therefore a controlled idealization, not a lower-quality substitute for 3D analysis.

This note develops three widely used reductions directly from three-dimensional isotropic elasticity:

- **Plane stress**, in which the stresses associated with the out-of-plane direction are negligible;
- **Plane strain**, in which the out-of-plane strain components are zero in the ideal model, or negligible for the accuracy required in a real body; and
- **Axisymmetric analysis**, in which the geometry and physical fields do not vary with the circumferential coordinate. Both torsion-free axisymmetry and axisymmetry with circumferential displacement are included.

These formulations are standard parts of classical and modern elasticity and mechanics-of-materials treatments; useful complementary developments are given by Barber, Timoshenko and Goodier, Boresi and Schmidt, Sadd, Ugural and Fenster, Landau and Lifshitz, and Slaughter [1, 2, 3, 4, 5, 6, 7].

For each formulation, the discussion begins with its physical meaning, derives its constitutive matrix from the 3D elastic law, identifies dependent stress or strain components, and closes with practical guidance on when the idealization is reliable and when it should be rejected.

Central principle. The best 2D formulation is selected by identifying what is physically negligible or invariant—not simply by inspecting whether the geometry looks thin, long, or round.

2 Common Three-Dimensional Elasticity Framework

2.1 Stress and strain conventions

For a small-strain, homogeneous, isotropic, linearly elastic material, use the engineering-shear convention

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \tau_{yz} & \tau_{xz} & \tau_{xy} \end{bmatrix}^{\mathsf{T}}, \tag{2.1}$$

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \gamma_{yz} & \gamma_{xz} & \gamma_{xy} \end{bmatrix}^{\mathsf{T}}. \tag{2.2}$$

The engineering shear strains satisfy

$$\gamma_{ij} = 2\varepsilon_{ij} \qquad (i \neq j). \tag{2.3}$$

The ordering used in a constitutive matrix must always match the ordering of the stress and strain vectors.

2.2 Tensorial and engineering shear strain

For the two-dimensional displacement field $u = u(x, y)$ and $v = v(x, y)$, the tensorial shear strain is

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{2.4}$$

whereas the engineering shear strain is

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2\varepsilon_{xy}. \tag{2.5}$$

Engineering shear strain measures the complete change in the angle between two material lines that were initially perpendicular. The symmetric strain tensor stores one-half of this angular change in each of its equal off-diagonal positions, ε_{xy} and ε_{yx}.

The symmetric and antisymmetric parts of the cross derivatives separate deformation from rigid rotation:

$$\varepsilon_{xy} = \frac{1}{2}\left(v_{,x} + u_{,y}\right), \qquad \omega_z = \frac{1}{2}\left(v_{,x} - u_{,y}\right). \tag{2.6}$$

For a small rigid rotation, $u = -\theta y$ and $v = \theta x$, so $\gamma_{xy} = 0$ while $\omega_z = \theta$. The body rotates without straining.

Convention warning. If γ_{xy} is used in the strain vector, then $\tau_{xy} = G\gamma_{xy}$. If tensorial ε_{xy} is used instead, then $\tau_{xy} = 2G\varepsilon_{xy}$. Mixing these conventions introduces a factor-of-two error in the shear response.

2.3 Three-dimensional isotropic stiffness matrix

Young's modulus E, Poisson's ratio ν, and shear modulus G are related by

$$G = \frac{E}{2(1+\nu)}. \tag{2.7}$$

The 3D constitutive law, developed in standard elasticity texts such as Refs. [1, 2, 4, 7], is

$$\boldsymbol{\sigma} = \mathbf{D}_{3D}\boldsymbol{\varepsilon}, \tag{2.8}$$

with

$$\mathbf{D}_{3D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}. \tag{2.9}$$

The corresponding normal compliance equations are

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right], \tag{2.10}$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right], \tag{2.11}$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right], \tag{2.12}$$

and the shear relations are $\gamma_{ij} = \tau_{ij}/G$.

3 Plane-Stress Analysis

3.1 Physical meaning and defining assumptions

Consider a thin body whose main surface lies in the x-y plane and whose thickness direction is z. If the two large faces normal to z are traction-free, the traction components on those faces are τ_{xz}, τ_{yz}, and σ_{zz}, and each is zero at the surfaces. For a sufficiently thin body under predominantly in-plane loading, these components remain small through most of the thickness. The resulting plane-stress reduction is also treated from a mechanics-of-materials perspective in Refs. [3, 5].

The ideal plane-stress conditions are

$$\boxed{\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0.} \tag{3.1}$$

The retained vectors are

$$\boldsymbol{\sigma}^{\mathrm{ps}} = \begin{bmatrix}\sigma_{xx} & \sigma_{yy} & \tau_{xy}\end{bmatrix}^{\mathsf{T}}, \qquad \boldsymbol{\varepsilon}^{\mathrm{ps}} = \begin{bmatrix}\varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy}\end{bmatrix}^{\mathsf{T}}. \tag{3.2}$$

Plane stress eliminates the out-of-plane *stresses*; it does not eliminate Poisson deformation through the thickness. In general, $\varepsilon_{zz} \neq 0$.

3.2 Reduction using the 3D compliance equations

Applying $\sigma_{zz} = 0$ to the 3D normal compliance equations gives

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}), \tag{3.3}$$

$$\varepsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}), \tag{3.4}$$

$$\varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}). \tag{3.5}$$

The transverse shear strains are zero for the isotropic elastic material because $\tau_{xz} = \tau_{yz} = 0$, while

$$\gamma_{xy} = \frac{\tau_{xy}}{G}. \tag{3.6}$$

Thus, the thickness strain remains as a dependent response caused by Poisson's effect. Under equal biaxial tension, $\sigma_{xx} = \sigma_{yy} = \sigma$, it becomes $\varepsilon_{zz} = -2\nu\sigma/E$; under pure in-plane shear it is zero.

The reduced compliance relationship is

$$\boldsymbol{\varepsilon}^{\mathrm{ps}} = \mathbf{S}_{\mathrm{ps}}\boldsymbol{\sigma}^{\mathrm{ps}}, \qquad \mathbf{S}_{\mathrm{ps}} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}. \tag{3.7}$$

3.3 Inversion and final constitutive matrix

Multiplying Eqs. (3.3)–(3.4) by E gives

$$E\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \end{bmatrix} = \begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix}\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{bmatrix}. \tag{3.8}$$

The determinant of the 2×2 matrix is $1 - \nu^2$, so

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \end{bmatrix}. \tag{3.9}$$

The shear relation may be written using the same leading factor:

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{1-\nu^2}\left(\frac{1-\nu}{2}\right)\gamma_{xy}. \tag{3.10}$$

Combining the normal and shear equations yields

$$\boxed{\boldsymbol{\sigma}^{\mathrm{ps}} = \mathbf{D}_{\mathrm{ps}}\boldsymbol{\varepsilon}^{\mathrm{ps}}, \qquad \mathbf{D}_{\mathrm{ps}} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}.} \tag{3.11}$$

3.4 Confirmation by elimination from the 3D stiffness law

The third normal row of Eq. (2.9) is

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{xx} + \nu\varepsilon_{yy} + (1-\nu)\varepsilon_{zz}\right]. \tag{3.12}$$

Enforcing $\sigma_{zz} = 0$ and solving for the dependent thickness strain gives

$$\boxed{\varepsilon_{zz} = -\frac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy}).} \tag{3.13}$$

Substituting Eq. (3.13) into the σ_{xx} equation gives

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} - \frac{\nu^2}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy})\right] \tag{3.14}$$

$$= \frac{E}{(1+\nu)(1-2\nu)}\frac{1-2\nu}{1-\nu}(\varepsilon_{xx} + \nu\varepsilon_{yy}) \tag{3.15}$$

$$= \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}). \tag{3.16}$$

The same substitution produces the σ_{yy} equation and confirms Eq. (3.11). The essential point is that ε_{zz} is eliminated by enforcing $\sigma_{zz} = 0$; it is not set to zero.

3.5 Kinematics and equilibrium

For the in-plane displacement field $u = u(x, y)$ and $v = v(x, y)$,

$$\varepsilon_{xx} = u_{,x}, \qquad \varepsilon_{yy} = v_{,y}, \qquad \gamma_{xy} = u_{,y} + v_{,x}. \tag{3.17}$$

Static equilibrium in the x-y plane requires

$$\sigma_{xx,x} + \tau_{xy,y} + b_x = 0, \tag{3.18}$$
$$\tau_{xy,x} + \sigma_{yy,y} + b_y = 0, \tag{3.19}$$

where b_x and b_y are body-force components per unit volume. On a boundary with outward normal $\boldsymbol{n} = [n_x, n_y]^\mathsf{T}$,

$$t_x = \sigma_{xx}n_x + \tau_{xy}n_y, \qquad t_y = \tau_{xy}n_x + \sigma_{yy}n_y. \tag{3.20}$$

3.6 When plane stress is a good approximation

Plane stress is most appropriate when the body is thin relative to its in-plane dimensions, the large faces are approximately traction-free, and loads and restraints act mainly in the plane. Geometry, material properties, and loading should not vary strongly through the thickness. Typical applications include

- thin sheets under in-plane tension, compression, or shear;
- sheet-metal components, thin webs, and gusset plates;
- membrane regions of thin-walled structures; and
- regions sufficiently far from edges, grips, concentrated loads, fasteners, welds, contacts, and abrupt thickness changes.

There is no universal thickness ratio that guarantees plane stress. The neglected components σ_{zz}, τ_{xz}, and τ_{yz} must be small relative to the quantities important to the analysis.

3.7 When plane stress should not be used

A conventional 2D plane-stress continuum should not be used when any of the following effects are important:

- **Loading normal to the plane.** Transverse pressure or force may produce out-of-plane displacement and bending, which the two-displacement continuum cannot represent.
- **Through-thickness normal or shear stress.** Contact compression, indentation, peel, interlaminar shear, and confined layers require the omitted components.
- **A body that is not sufficiently thin**, or a response that varies strongly through the thickness.
- **Local three-dimensional regions** near supports, holes, notches, fasteners, welds, free edges, contacts, or load-introduction points.
- **Layered structures** when delamination or interface stresses control the response.

Plane-stress continuum versus plate or shell behavior. Plate and shell theories may use a plane-stress *material law* while also including transverse displacement, rotation, and bending curvature. A thin plate under normal pressure may therefore be suitable for plate or shell theory, but not for a 2D plane-stress continuum drawn in the plane of the plate.

4 Plane-Strain Analysis

4.1 Physical meaning and kinematic assumptions

Consider a body whose cross-section lies in the x-y plane and which extends a long distance in the z direction. Standard plane strain idealizes an interior region whose geometry, material, loading, and restraints are effectively independent of z. Classical discussions of this interior-region idealization and its constitutive consequences are available in Refs. [2, 3, 4]. The ideal displacement field may be written

$$u = u(x, y), \qquad v = v(x, y), \qquad w = 0, \tag{4.1}$$

up to an arbitrary rigid translation in z. Consequently,

$$\boxed{\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0.} \tag{4.2}$$

In a real engineering body, these components need not be exactly zero. The plane-strain approximation is appropriate when the out-of-plane strains are negligible relative to the important in-plane strains and do not materially affect the results of interest.

The retained strain vector is

$$\boldsymbol{\varepsilon}^{\text{pe}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} \end{bmatrix}^{\mathsf{T}}. \tag{4.3}$$

For the isotropic material considered here, $\tau_{xz} = \tau_{yz} = 0$, but the normal reaction stress σ_{zz} is generally nonzero.

> Plane strain suppresses the out-of-plane *strain*; it does not eliminate the out-of-plane stress. The material attempts to undergo Poisson deformation in z, and the constraint produces σ_{zz}.

4.2 Direct reduction from the 3D stiffness equations

Define

$$C = \frac{E}{(1+\nu)(1-2\nu)}. \tag{4.4}$$

The three normal rows of the 3D stiffness law are

$$\sigma_{xx} = C\left[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}\right], \tag{4.5}$$

$$\sigma_{yy} = C\left[\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy} + \nu\varepsilon_{zz}\right], \tag{4.6}$$

$$\sigma_{zz} = C\left[\nu\varepsilon_{xx} + \nu\varepsilon_{yy} + (1-\nu)\varepsilon_{zz}\right]. \tag{4.7}$$

Applying $\varepsilon_{zz} = 0$ gives

$$\sigma_{xx} = C\left[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}\right], \tag{4.8}$$

$$\sigma_{yy} = C\left[\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy}\right], \tag{4.9}$$

$$\sigma_{zz} = C\nu(\varepsilon_{xx} + \varepsilon_{yy}). \tag{4.10}$$

The in-plane shear relation is

$$\tau_{xy} = G\gamma_{xy} = C\left(\frac{1-2\nu}{2}\right)\gamma_{xy}. \tag{4.11}$$

The final in-plane constitutive law is therefore

$$\boxed{\boldsymbol{\sigma}^{\text{pe}} = \mathbf{D}_{\text{pe}}\boldsymbol{\varepsilon}^{\text{pe}}, \qquad \mathbf{D}_{\text{pe}} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix},} \tag{4.12}$$

where

$$\boldsymbol{\sigma}^{\text{pe}} = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \tau_{xy} \end{bmatrix}^{\mathsf{T}}. \tag{4.13}$$

4.3 Recovery and interpretation of the out-of-plane stress

Equation (4.10) gives

$$\boxed{\sigma_{zz} = \frac{E\nu}{(1+\nu)(1-2\nu)}(\varepsilon_{xx} + \varepsilon_{yy}).} \tag{4.14}$$

Adding Eqs. (4.8) and (4.9) produces $\sigma_{xx} + \sigma_{yy} = C(\varepsilon_{xx} + \varepsilon_{yy})$; hence

$$\boxed{\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}).} \tag{4.15}$$

Under in-plane extension, the material attempts to contract in z, and a tensile reaction stress prevents that contraction. Under in-plane compression, a compressive reaction stress prevents Poisson expansion. Pure in-plane shear produces no σ_{zz} for the linear isotropic material.

4.4 Confirmation using the 3D compliance equations

The condition $\varepsilon_{zz} = 0$ applied to the 3D compliance law gives

$$0 = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right], \tag{4.16}$$

which immediately confirms Eq. (4.15). Substitution into the remaining compliance equations gives

$$\varepsilon_{xx} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right], \tag{4.17}$$

$$\varepsilon_{yy} = \frac{1+\nu}{E}\left[-\nu\sigma_{xx} + (1-\nu)\sigma_{yy}\right], \tag{4.18}$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}. \tag{4.19}$$

Thus,

$$\boldsymbol{\varepsilon}^{\mathrm{pe}} = \mathbf{S}_{\mathrm{pe}}\boldsymbol{\sigma}^{\mathrm{pe}}, \qquad \mathbf{S}_{\mathrm{pe}} = \frac{1+\nu}{E}\begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix}, \tag{4.20}$$

whose inverse is Eq. (4.12).

4.5 Why plane strain is more constrained

Plane stress permits Poisson contraction or expansion in the out-of-plane direction. Plane strain suppresses that deformation and generates σ_{zz}. The added constraint generally makes the in-plane normal response appear stiffer than under plane stress. The difference becomes especially strong as ν approaches 0.5, because resistance to volumetric deformation increases substantially.

This is a physical confinement effect, not merely an algebraic difference between two matrices. It affects displacements, hydrostatic stress, yielding, and pressure-sensitive failure.

4.6 Kinematics, equilibrium, and complete stress evaluation

The in-plane kinematics remain

$$\varepsilon_{xx} = u_{,x}, \qquad \varepsilon_{yy} = v_{,y}, \qquad \gamma_{xy} = u_{,y} + v_{,x}, \tag{4.21}$$

and equilibrium requires

$$\sigma_{xx,x} + \tau_{xy,y} + b_x = 0, \tag{4.22}$$

$$\tau_{xy,x} + \sigma_{yy,y} + b_y = 0. \tag{4.23}$$

Although equilibrium is solved in two dimensions, the nonzero stress state is

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}. \tag{4.24}$$

The two in-plane principal stresses are

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}, \tag{4.25}$$

and σ_{zz} is the third principal stress. The 3D von Mises stress is

$$\sigma_{\mathrm{vm}} = \sqrt{\frac{1}{2}\left[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2\right] + 3\tau_{xy}^2}. \tag{4.26}$$

Discarding σ_{zz} would incorrectly evaluate the state as plane stress.

4.7 When plane strain is a good approximation

Plane strain is most appropriate for an interior cross-section of a long or longitudinally constrained body when

- the cross-section, material, loading, and restraints vary negligibly along z;
- the region of interest is far enough from the ends that end effects have decayed;
- ε_{zz}, γ_{xz}, and γ_{yz} are negligible for the results and accuracy required; and
- the important response occurs in the x-y cross-section.

Typical applications include interior sections of long dams, retaining walls, embankments, tunnels, underground openings, long seals, and prismatic bodies whose longitudinal deformation is restrained.

A long body is not automatically in plane strain. Length does not itself enforce $\varepsilon_{zz} = 0$. A long body that is free to expand because of Poisson's effect, temperature, pressure, or axial force may require generalized plane strain or a three-dimensional description.

4.8 When plane strain should not be used

Standard plane strain is unreliable when

- the body is not long relative to its cross-sectional dimensions;
- the region of interest lies near a free or loaded end;
- geometry, material, loading, restraint, contact, or temperature varies with z;
- longitudinal expansion, contraction, bending, torsion, or warping is important;
- localized three-dimensional features control the desired result; or
- the body requires a nonzero average longitudinal strain.

4.9 Generalized plane strain

Some long bodies undergo an approximately uniform but nonzero longitudinal strain

$$\varepsilon_{zz} = \varepsilon_0 \neq 0. \tag{4.27}$$

A representative displacement field is

$$u = u(x, y), \qquad v = v(x, y), \qquad w = \varepsilon_0 z. \tag{4.28}$$

This *generalized plane-strain* condition is useful when an average axial strain or resultant axial force is prescribed, as in some thermally expanding, pressurized, or axially loaded long bodies. If longitudinal strain varies significantly across the cross-section or along z, a 3D description is required.

5 Axisymmetric Analysis

5.1 Cylindrical coordinates and the meaning of axisymmetry

Axisymmetric analysis uses cylindrical coordinates (r, θ, z), where r is radial, θ is circumferential, and z is axial. A point in the r-z cross-section represents a complete circular ring in the three-dimensional body. Revolving a differential area $\mathrm{d}A$ through 360° creates the differential volume

$$\mathrm{d}V = 2\pi r \,\mathrm{d}A. \tag{5.1}$$

The cylindrical-coordinate elasticity relations and their axisymmetric reduction are developed in Refs. [1, 2, 4, 7].

In the ideal model, axisymmetry means that the geometry and relevant physical fields are independent of θ:

$$\boxed{\frac{\partial(\,\cdot\,)}{\partial\theta} = 0.} \tag{5.2}$$

In a real body, circumferential variations may not vanish exactly; the approximation is appropriate when they are negligible for the desired results. Equation (5.2) concerns *variation with* θ, not deformation *in the* θ direction. Hoop strain and hoop stress may be large.

5.2 Hoop strain

A circular material line of radius r has circumference $2\pi r$. After a small radial displacement u_r, its circumference becomes $2\pi(r + u_r)$, so

$$\varepsilon_\theta = \frac{2\pi u_r}{2\pi r} = \boxed{\frac{u_r}{r}}. \tag{5.3}$$

Thus, a uniformly expanding cylinder is axisymmetric even though $\varepsilon_\theta \neq 0$ and $\sigma_\theta \neq 0$. Axisymmetry eliminates circumferential variation, not circumferential response.

5.3 Standard torsion-free axisymmetric kinematics

The standard torsion-free displacement field is

$$u_r = u_r(r, z), \qquad u_z = u_z(r, z), \qquad u_\theta = 0. \tag{5.4}$$

The four active engineering strains are

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \tag{5.5}$$

$$\varepsilon_z = \frac{\partial u_z}{\partial z}, \tag{5.6}$$

$$\varepsilon_\theta = \frac{u_r}{r}, \tag{5.7}$$

$$\gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}. \tag{5.8}$$

Using the order (r, z, θ, rz),

$$\boldsymbol{\varepsilon}^{\text{axi}} = \begin{bmatrix} \varepsilon_r & \varepsilon_z & \varepsilon_\theta & \gamma_{rz} \end{bmatrix}^\mathsf{T}, \qquad \boldsymbol{\sigma}^{\text{axi}} = \begin{bmatrix} \sigma_r & \sigma_z & \sigma_\theta & \tau_{rz} \end{bmatrix}^\mathsf{T}. \tag{5.9}$$

The stress tensor, displayed in the physical basis (r, θ, z), is

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_r & 0 & \tau_{rz} \\ 0 & \sigma_\theta & 0 \\ \tau_{rz} & 0 & \sigma_z \end{bmatrix}. \tag{5.10}$$

5.4 The standard 4 by 4 constitutive matrix

The 3D isotropic normal relations written in cylindrical coordinates are

$$\sigma_r = C\left[(1-\nu)\varepsilon_r + \nu\varepsilon_z + \nu\varepsilon_\theta\right], \tag{5.11}$$

$$\sigma_z = C\left[\nu\varepsilon_r + (1-\nu)\varepsilon_z + \nu\varepsilon_\theta\right], \tag{5.12}$$

$$\sigma_\theta = C\left[\nu\varepsilon_r + \nu\varepsilon_z + (1-\nu)\varepsilon_\theta\right], \tag{5.13}$$

$$\tau_{rz} = G\gamma_{rz} = C\left(\frac{1-2\nu}{2}\right)\gamma_{rz}, \tag{5.14}$$

where $C = E/[(1+\nu)(1-2\nu)]$. Therefore,

$$\boxed{\boldsymbol{\sigma}^{\text{axi}} = \mathbf{D}_{\text{axi}}\boldsymbol{\varepsilon}^{\text{axi}}, \qquad \mathbf{D}_{\text{axi}} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}.} \tag{5.15}$$

All three normal strains remain active, and the general meridional displacement field can produce γ_{rz}. This is why the standard torsion-free axisymmetric solid uses a 4×4 matrix rather than a 3×3 matrix.

5.5 Axisymmetric equilibrium

Radial and axial equilibrium require

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + b_r = 0, \tag{5.16}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + b_z = 0. \tag{5.17}$$

The curvature term $(\sigma_r - \sigma_\theta)/r$ has no counterpart in Cartesian plane strain. It represents circumferential force balance and is one reason a cylindrical body cannot generally be replaced by a Cartesian plane-strain model.

For a boundary in the r-z plane with outward normal $\boldsymbol{n} = [n_r, n_z]^\mathsf{T}$,

$$t_r = \sigma_r n_r + \tau_{rz} n_z, \qquad t_z = \tau_{rz} n_r + \sigma_z n_z. \tag{5.18}$$

5.6 Regularity at the axis

Equation (5.3) appears singular at $r = 0$. A physically regular field must satisfy

$$u_r = 0 \qquad \text{at } r = 0, \tag{5.19}$$

and u_r must approach zero smoothly. Then

$$\lim_{r \to 0} \frac{u_r}{r} = \left.\frac{\partial u_r}{\partial r}\right|_{r=0}, \tag{5.20}$$

so the hoop strain remains finite. Geometry and loading at the axis must likewise be regular; a nonzero radial displacement at $r = 0$ is physically inadmissible.

5.7 Axisymmetry with torsion or circumferential displacement

Axisymmetry does not require $u_\theta = 0$. A circular shaft may twist while every quantity remains independent of θ. The more general displacement field is

$$u_r = u_r(r, z), \qquad u_z = u_z(r, z), \qquad u_\theta = u_\theta(r, z), \tag{5.21}$$

with Eq. (5.2) still satisfied. The circumferential displacement introduces two additional engineering shear strains:

$$\gamma_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, \tag{5.22}$$

$$\gamma_{z\theta} = \frac{\partial u_\theta}{\partial z}. \tag{5.23}$$

The complete vectors, ordered as $(r, z, \theta, rz, r\theta, z\theta)$, are

$$\boldsymbol{\varepsilon}^{\text{axit}} = \begin{bmatrix} \varepsilon_r & \varepsilon_z & \varepsilon_\theta & \gamma_{rz} & \gamma_{r\theta} & \gamma_{z\theta} \end{bmatrix}^{\mathsf{T}}, \tag{5.24}$$

$$\boldsymbol{\sigma}^{\text{axit}} = \begin{bmatrix} \sigma_r & \sigma_z & \sigma_\theta & \tau_{rz} & \tau_{r\theta} & \tau_{z\theta} \end{bmatrix}^{\mathsf{T}}. \tag{5.25}$$

5.7.1 Rigid rotation and shaft torsion

If $u_\theta = \Omega r$ with constant Ω, then

$$\gamma_{r\theta} = \Omega - \Omega = 0, \qquad \gamma_{z\theta} = 0, \tag{5.26}$$

so the body undergoes rigid rotation without strain. For a circular shaft whose cross-section rotates through $\phi(z)$,

$$u_\theta = r\phi(z), \tag{5.27}$$

which gives

$$\gamma_{r\theta} = 0, \qquad \gamma_{z\theta} = r\frac{\mathrm{d}\phi}{\mathrm{d}z}. \tag{5.28}$$

The familiar linear variation of torsional shear strain with radius is recovered.

5.8 The 6 by 6 constitutive matrix with torsion

For isotropic linear elasticity,

$$\tau_{r\theta} = G\gamma_{r\theta}, \qquad \tau_{z\theta} = G\gamma_{z\theta}. \tag{5.29}$$

Together with the four torsion-free relations, the constitutive law becomes

$$\boxed{\mathbf{D}_{\text{axit}} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}}. \tag{5.30}$$

Thus,

$$\boldsymbol{\sigma}^{\text{axit}} = \mathbf{D}_{\text{axit}}\boldsymbol{\varepsilon}^{\text{axit}}. \tag{5.31}$$

Relation to the general 3D constitutive matrix. For an isotropic linear-elastic material, $\mathbf{D}_{\mathrm{axi}t}$ is not a new material matrix. It is exactly the standard 6×6 three-dimensional isotropic elasticity matrix, written in cylindrical components and reordered to match $(r, z, \theta, rz, r\theta, z\theta)$. Axisymmetry imposes $\partial(\cdot)/\partial\theta = 0$ on the geometry and physical fields; it does not reduce the three-dimensional constitutive behavior. When $u_\theta = 0$, the last two shear strain–stress pairs become inactive, leaving the 4×4 torsion-free block in Eq. (5.15).

For isotropic linear elasticity, the two additional torsional shear responses are uncoupled from the 4×4 meridional block. Material anisotropy, nonlinear kinematics, contact, or inelastic response may introduce additional coupling.

The circumferential equilibrium equation is

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{z\theta}}{\partial z} + \frac{2\tau_{r\theta}}{r} + b_\theta = 0. \tag{5.32}$$

Regularity requires $u_\theta = 0$ at $r = 0$ and a smooth approach proportional to r.

5.9 Special reduced axisymmetric forms

The constitutive-matrix size is determined by the active kinematics, not by the word "axisymmetric" alone.

5.9.1 Restricted normal 3 by 3 form

If an additional condition guarantees $\gamma_{rz} = \tau_{rz} = 0$, only the three normal components remain:

$$\begin{bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \end{bmatrix}. \tag{5.33}$$

This is a restricted subclass, not the general axisymmetric solid.

5.9.2 Thin axisymmetric disk under plane stress

For a thin disk with radial displacement $u_r = u_r(r)$ and negligible σ_z, the active normal components may reduce to

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{bmatrix}, \qquad \varepsilon_r = \frac{\mathrm{d}u_r}{\mathrm{d}r}, \quad \varepsilon_\theta = \frac{u_r}{r}. \tag{5.34}$$

This is simultaneously axisymmetric and plane stress. Axisymmetry describes invariance with θ; plane stress describes the condition through the disk thickness.

5.10 When axisymmetric analysis is appropriate

Standard torsion-free axisymmetry is well suited to bodies of revolution whose material, loading, restraints, contact, and temperature are circumferentially invariant. Examples include

- thick cylinders, pipes, tanks, and pressure vessels under uniform pressure;
- concentric press fits, pistons, O-rings, seals, and coaxial contact;
- circular diaphragms, centered indentation, and circular foundations;
- axisymmetric forming, thermal loading, rotating disks, and flywheels; and
- geometries that vary strongly with r and z but remain continuous around the circumference, such as end caps, fillets, and support rings.

Axisymmetry with torsion is appropriate when u_θ is present but all fields remain independent of θ. Applications include circular shafts and tubes under distributed torque, combined pressure and torsion, or tangential traction applied continuously around a circular interface.

Torque must also be axisymmetric. A continuous circumferential traction or twist can be represented. Torque introduced through one key, several bolts, discrete gear teeth, or localized lugs is not axisymmetric near those features, even if the net resultant torque is the same.

5.11 When axisymmetric analysis should not be used

Axisymmetry is invalid when an important quantity varies significantly with θ. Common violations include

- side holes, side nozzles, keyways, spokes, blades, gear teeth, lugs, localized cracks, or nonuniform wall thickness;
- point forces, eccentric loads, transverse bending, partial pressure, or one tangential force;
- discrete bolts, feet, clamps, supports, welds, or bearings;
- partial contact, one-sided contact, misalignment, or circumferentially varying friction;
- temperature or material properties that vary around the circumference;
- gravity acting transverse to the axis, as in a horizontal cylinder, when its circumferential variation matters; and
- torsion applied through discrete features whose local stresses are required.

Even when the initial geometry and loading are axisymmetric, the actual response may not remain so. Lobar buckling, ovalization, circumferential wrinkling, localized yielding, fracture, damage, or contact separation may break the symmetry. An axisymmetric model constrains the response to remain axisymmetric and therefore cannot predict a non-axisymmetric mode.

Gravity example. Gravity parallel to the z axis is axisymmetric. Gravity perpendicular to the axis generally is not, because its radial and circumferential components change with θ.

5.12 Axisymmetry versus plane strain

Both formulations use a two-dimensional cross-section, but they represent different three-dimensional bodies. Plane strain represents a prismatic body extending in a straight out-of-plane direction and suppresses the corresponding strain. Axisymmetry represents a cross-section revolved about an axis and retains

$$\varepsilon_\theta = \frac{u_r}{r}, \qquad \sigma_\theta \neq 0 \text{ in general.} \tag{5.35}$$

It also contains the curvature terms in Eqs. (5.16)–(5.17). Using Cartesian plane strain for a cylindrical cross-section generally omits hoop strain, hoop stress, curvature, and circumferential equilibrium.

6 Comparison and Model-Selection Guide

6.1 What each idealization removes or preserves

6.2 Constitutive matrices at a glance

With engineering shear strain, plane stress uses

$$\mathbf{D}_{\text{ps}} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}, \tag{6.1}$$

Table 1: Core distinctions among the principal 2D elasticity idealizations.

Formulation	Defining condition	Important retained response
Plane stress	$\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$	ε_{zz} is generally nonzero because thickness contraction or expansion is free.
Plane strain	$\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$ in the ideal model	σ_{zz} is generally nonzero because it enforces the out-of-plane constraint.
Axisymmetric solid	$\partial(\cdot)/\partial\theta = 0$ and $u_\theta = 0$	Hoop strain $\varepsilon_\theta = u_r/r$ and hoop stress σ_θ remain active.
Axisymmetric with torsion	$\partial(\cdot)/\partial\theta = 0$ with $u_\theta \neq 0$ permitted	$\gamma_{r\theta}$, $\gamma_{z\theta}$, $\tau_{r\theta}$, and $\tau_{z\theta}$ are added.

whereas plane strain uses

$$\mathbf{D}_{\text{pe}} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}. \tag{6.2}$$

The standard axisymmetric solid uses

$$\mathbf{D}_{\text{axi}} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}. \tag{6.3}$$

Axisymmetry with torsion adds two shear components and uses Eq. (5.30).

6.3 A disciplined selection process

Before accepting any 2D idealization:

1. **Define the omitted coordinate or symmetry direction.** State clearly whether the model neglects stress, neglects strain, or assumes circumferential invariance.
2. **Check the complete physical problem.** Geometry alone is insufficient; loading, support, material, contact, temperature, and expected deformation must satisfy the same idealization.
3. **Identify local violations.** Ends, free edges, contacts, fasteners, notches, discrete supports, and load-introduction regions commonly produce three-dimensional behavior.
4. **Match the assumption to the desired result.** A 2D model may predict global stiffness accurately while missing a local stress concentration or a symmetry-breaking failure mode.
5. **Recover dependent components.** Calculate ε_{zz} in plane stress, σ_{zz} in plane strain, and hoop or torsional components in axisymmetry before evaluating the complete response.
6. **Verify when consequences are important.** Compare with analytical limits, experimental evidence, or a targeted 3D calculation when the validity of the idealization materially affects design decisions.

> The question is not whether a 2D model is "accurate" in the abstract. The question is whether its neglected components are small enough for the specific quantity, location, and decision that the analysis must support.

7 Concluding Remarks

Plane stress, plane strain, and axisymmetric elasticity are rigorous reductions of the three-dimensional continuum equations when their defining assumptions are satisfied. Their constitutive matrices differ because they preserve different parts of the 3D response.

Plane stress sets the out-of-plane stresses to zero and permits Poisson deformation through the thickness. Plane strain suppresses the out-of-plane strains and generates a reaction stress σ_{zz}. Axisymmetry preserves the three normal directions but removes circumferential variation; it therefore retains hoop strain and hoop stress. When circumferential displacement is admitted, two torsional shear components extend the standard 4×4 axisymmetric law to a 6×6 form.

The efficiency of a two-dimensional analysis comes from a physical assumption, and that assumption establishes the boundary of its validity. Thinness does not guarantee plane stress, length does not guarantee plane strain, and roundness does not guarantee axisymmetry. Careful selection, recovery of dependent components, and attention to local three-dimensional behavior are what make a 2D analysis trustworthy.

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